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Quantum complex Minkowski space

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Abstract

The complex Minkowski phase space has the physical interpretation of the phase space of the scalar massive conformal particle. The aim of the paper is the construction and investigation of the quantum complex Minkowski space.

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1. Introduction

Extending the Poincaré group by dilation and acceleration transformations, one obtains the conformal group $\mathrm{SU}(2, 2)/\mathbb{Z}_4$, which is the symmetry group of the conformal structure of compactified Minkowski space-time M, where $\mathbb{Z}_4 = \{i^k \mathrm{id} : k = 0, 1, 2, 3\}$ is the centralizer of $\mathrm{SU}(2, 2)$. According to the prevailing point of view $\mathrm{SU}(2, 2)/\mathbb{Z}_4$ is the symmetry group for physical models which describe massless fields or particles, but has no application to the theory of massive objects. However, using the twistor description [1] of Minkowski space-time and the orbit method [2], the different orbits of $\mathrm{SU}(2, 2)/\mathbb{Z}_4$ in the conformally compactified complex Minkowski space $\mathbb{M} := M^{\mathbb{C}}$ may be considered to be the classical phase spaces of massless and massive scalar conformal particles, antiparticles and tachyons, see [3,4].

The motivation for various attempts to construct models of non-commutative Minkowski space-time is the belief that this is the proper way to avoid divergences in quantum field theory [5]. Here, on the other hand, our aim is to quantize the classical phase space $\mathbb{M}^{++} \subset \mathbb{M}$ of the massive particle by replacing it by the Toeplitz-like operator C^*-algebra $\mathcal{M}^{++}$. To this end we first quantize the

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classical states of the massive scalar conformal particle by constructing the coherent state map $\mathcal{K} : \mathbb{M}^{++} \to \mathbb{CP}(\mathcal{H})$ of $\mathbb{M}^{++}$ into the complex projective Hilbert space $\mathbb{CP}(\mathcal{H})$, i.e. the space of the pure states of the system. In the next step we define the Banach algebra $\overline{\mathcal{P}}^{++}$ of annihilation operators as the ones having the coherent states $\mathcal{K}(m)$, $m \in \mathbb{M}^{++}$, as eigenvectors. Finally, the quantum phase space $\mathcal{M}^{++}$ will be the C^*-algebra generated by $\overline{\mathcal{P}}^{++}$.

Let us remark that application of the above method of quantization to the case of $\mathbb{R}^{2N}$ phase space leads to the Heisenberg–Weyl algebra. In our construction the conformal group and $\mathcal{M}^{++}$ are related in exactly the same way as are the Heisenberg group and the Heisenberg–Weyl algebra.

The conformally compactified Minkowski space M can be reconstructed from $\mathcal{M}^{++}$ as the Šilov boundary of the interior of the spectrum of the commutative Banach algebra $\mathcal{P}^{++}$. It can also be considered in the framework of Kostant–Souriau quantization as the $\mathrm{SU}(2,2)/\mathbb{Z}_4$-invariant configuration space for the phase space T^*M. Similarly, if we consider the holomorphic model [4], see Sections 2 and 3, the classical conformal phase space $\mathbb{M}^{++}$ has the interpretation of the configuration space constructed by the $\mathrm{SU}(2,2)/\mathbb{Z}_4$-invariant Kähler polarization. In [4] a model of the classical field theory on $\mathbb{M}^{++}$ was proposed. This paper is an effort, developing the results presented in [6], to construct a quantum description of the conformal massive particle, see Section 4. In Section 5 the physical interpretation of the quantum phase space $\mathcal{M}^{++}$ is discussed.

2. Complex Minkowski space as the phase space of the conformal scalar massive particle

Following [3,7,4], we present the twistor description of phase spaces of the conformal scalar massive particles. Let us recall that twistor space $\mathbf{T}$ is $\mathbb{C}^4$ equipped with a Hermitian form η of signature $(++--)$. The symmetry group of $\mathbf{T}$ is the group $\mathrm{SU}(2,2)$, where $g \in \mathrm{SU}(2,2)$ iff $g^\dagger \eta g = \eta$ and $\det g = 1$.

In relativistic mechanics the elementary phase spaces are given by the coadjoint orbits of the Poincaré group, see [8], which are parametrized in this case by mass, spin, and signature of the energy of the relativistic particle. Similarly, elementary phase spaces for conformal group one identifies with its coadjoint orbits. Since conformal Lie algebra su(2, 2) is simple we will identify its dual su(2, 2)* with su(2, 2) using Cartan–Killing form:

$$\langle X, Y\rangle = \tfrac{1}{2}\mathrm{Tr}(XY), \tag{2.1}$$

where $X, Y \in \mathrm{su}(2,2)$. Thus the coadjoint representation $\mathrm{Ad}^*_g : \mathrm{su}(2,2)^* \to \mathrm{su}(2,2)^*$ is identified with the adjoint one

$$\mathrm{Ad}_g X = gXg^{-1}, \tag{2.2}$$

where $g \in \mathrm{SU}(2,2)$. For the complete description and physical interpretation of $\mathrm{Ad}^*(\mathrm{SU}(2,2))$-orbits see [9,7].

One defines the compactified complex Minkowski space $\mathbb{M}$ as the Grassmannian of two-dimensional complex vector subspaces $w \subset \mathbf{T}$ of the twistor space and $\mathrm{SU}(2,2)$ acts on $\mathbb{M}$ by

$$\sigma_g : w \to gw. \tag{2.3}$$

The Grassmannian $\mathbb{M}$ splits into the orbits $\mathbb{M}^{kl}$ indexed by the signatures of the restricted Hermitian forms $\mathrm{sign}\,\eta|_z = (k, l)$, where $k, l = +, -, 0$.

The orbit $\mathbb{M}^{00}$ consisting of subspaces isotropic with respect to η is the conformal compactification M of real Minkowski space and $\mathbb{M}$ is the complexification of $M = \mathbb{M}^{00}$.

The cotangent bundle $T^*\mathbb{M}^{00} \to \mathbb{M}^{00}$ is isomorphic with the vector bundle $\{(x, X) \in \mathbb{M}^{00} \times \mathrm{su}(2,2) : \mathrm{im}X \subset x \subset \ker X\} =: \mathbb{N} \overset{pr_1}{\to} \mathbb{M}^{00}$, where pr_1 is the projection on the first component of the product $\mathbb{M}^{00} \times \mathrm{su}(2,2)$. The vector bundle isomorphism $T^*\mathbb{M}^{00} \cong \mathbb{N}$ is defined by the following sequence $T^*_x\mathbb{M}^{00} \cong (\mathrm{su}(2,2)/\mathrm{su}(2,2)_x)^* \cong \{X \in \mathrm{su}(2,2) : \mathrm{Tr}\, YX = 0; \forall\, Y \in \mathrm{su}(2,2)_x\} \cong \{X \in \mathrm{su}(2,2) : \mathrm{im}X \subset x \subset \ker X\} = pr_1^{-1}(x)$ of the vector space isomorphisms.

There exists a conformal structure on $\mathbb{N}$ defined by the cones $C_x := \{X \in pr_1^{-1}(x) : \dim_{\mathbb{R}} \mathrm{im}\, X \leq 1\} \subset pr_1^{-1}(x) \cong T^*_x\mathbb{M}^{00}$, $x \in \mathbb{M}^{00}$. This conformal structure is invariant with respect to the action of $\mathrm{SU}(2,2)$ on $\mathbb{N}$ defined by

$$\alpha_g : (x, X) \mapsto (gx, gXg^{-1}) \tag{2.4}$$

for $g \in \mathrm{SU}(2,2)$.

The eight-dimensional orbits of the action (2.4) are: the bundle $\mathbb{N}^{++} \to \mathbb{M}^{00}$ of upper halves of the interiors of the cones, the bundle $\mathbb{N}^{--} \to \mathbb{M}^{00}$ of bottom halves of the interiors of the cones and the bundle $\mathbb{N}^{+-} \to \mathbb{M}^{00}$ of exteriors of the cones.

Similarly, the action (2.3) of $\mathrm{SU}(2,2)$ on $\mathbb{M}$ generates three eight-dimensional orbits: $\mathbb{M}^{++}$, $\mathbb{M}^{--}$ and $\mathbb{M}^{+-}$.

One has maps $J_0 : \tilde{\mathbb{N}} \to \mathrm{su}(2,2)$ and $J_\lambda : \tilde{\mathbb{M}} \to \mathrm{su}(2,2)$ of $\tilde{\mathbb{N}} := \mathbb{N}^{++} \cup \mathbb{N}^{--} \cup \mathbb{N}^{+-}$ and $\tilde{\mathbb{M}} := \mathbb{M}^{++} \cup \mathbb{M}^{--} \cup \mathbb{M}^{+-}$ into $\mathrm{su}(2,2)$ defined by:

$$J_0(x, X) := X, \tag{2.5}$$

$$J_\lambda(w) := i\lambda(\pi_w - \pi_{w^\perp}), \tag{2.6}$$

where $\perp: \tilde{\mathbb{M}} \mapsto \tilde{\mathbb{M}}$ is the involution, which maps $w \in \tilde{\mathbb{M}}$ on its orthogonal complement $w^\perp$ (with respect to the twistor forms η) and $\pi_w : \mathbf{T} \mapsto \mathbf{T}$ and $\pi_{w^\perp} : \mathbf{T} \mapsto \mathbf{T}$ are the projections defined by the decomposition $\mathbf{T} = w \oplus w^\perp$.

The maps J_0 and J_λ are equivariant with respect to the actions α and σ respectively and Ad-action of the conformal group. Thus J_0 maps $\mathbb{N}^{++}$, $\mathbb{N}^{+-}$, $\mathbb{N}^{--}$ on the eight-dimensional nilpotent Ad-orbits and J_λ maps $\mathbb{M}^{++}$, $\mathbb{M}^{++}$, $\mathbb{M}^{++}$ on the eight-dimensional simple Ad-orbits which consist of $X \in \mathrm{su}(2,2)$ with eigenvalues $i\lambda$ and $-i\lambda$. Using the Kirillov construction [2] we obtain the conformally invariant symplectic form ω_0 on $\tilde{\mathbb{N}}$ (identical with the canonical symplectic form of $T^*\mathbb{M}^{00}$) and the conformally invariant Kähler form ω_λ on $\tilde{\mathbb{M}}$. So $(\tilde{\mathbb{N}}, \omega_0)$ and $(\tilde{\mathbb{M}}, \omega_\lambda)$ are eight-dimensional conformal symplectic manifolds with momentum maps given by (2.5), (2.6).

In order to show that $\tilde{\mathbb{N}}$ and $\tilde{\mathbb{M}}$ have a physical interpretation of the phase spaces of the conformal scalar massive particles, let us take the coordinate description of the presented models. We fix an element $\infty \in \mathbb{M}^{00}$, called point at infinity. One defines the Minkowski space $\mathbb{M}^{00}_\infty$ as the affine space of elements $w \in \mathbb{M}^{00}$ which are transversal to ∞, i.e. $w \oplus \infty = \mathbf{T}$. The elements $w \in \mathbb{M}^{00}$ which intersect with ∞ in more than one-dimension, i.e. $\dim_{\mathbb{C}}(w \cap \infty) \geq 1$, form a cone C_∞ at infinity, so

$$\mathbb{M}^{00} = \mathbb{M}^{00}_\infty \cup C_\infty \cong \mathbb{S}^1 \times \mathbb{S}^3.$$

The cones $C_x = \{x' \in \mathbb{M}^{00} : \dim_{\mathbb{C}}(x \cap x') \geq 1\}$ define a conformal structure on $\mathbb{M}^{00}$, invariant with respect to the conformal group action given by (2.3). The Poincaré group $\mathbf{P}_\infty$ extended by the dilations is defined as the stabilizer $(\mathrm{SU}(2,2)/\mathbb{Z}_4)_\infty$ of the element ∞. The intersections of the stabilizers $(\mathrm{SU}(2,2)/\mathbb{Z}_4)_\infty \cap (\mathrm{SU}(2,2)/\mathbb{Z}_4)_0$, where $0 \in \mathbb{M}^{00}_\infty$ is the origin of the inertial coordinates system, is the Lorentz group extended by dilations. One defines the Lorentz group

$\mathbf{L}_{0,\infty}$ and the group of dilations $\mathbf{D}_{0,\infty}$ respectively as the commutator and the centralizer of $(\mathrm{SU}(2,2)/\mathbb{Z}_4)_\infty \cap (\mathrm{SU}(2,2)/\mathbb{Z}_4)_0$ respectively. Finally, the group of Minkowski space translations $\mathbf{T}_\infty$ consists of the elements $\exp X$, where $X \in \mathrm{su}(2,2)$ satisfies $\mathrm{im}\, X \subset \infty \subset \ker X$, while the elements $\exp X$ fulfilling $\mathrm{im}\, X \subset \mathbf{0} \subset \ker X$, define the commutative subgroup $\mathbf{A}_0$ of four-accelerations.

Let us assume in the following that

$$\eta = i\begin{pmatrix} 0 & \sigma_0 \\ -\sigma_0 & 0 \end{pmatrix}, \qquad \infty = \left\{ \begin{pmatrix} \zeta \\ 0 \end{pmatrix} : \zeta \in \mathbb{C}^2 \right\}, \qquad \mathbf{0} = \left\{ \begin{pmatrix} 0 \\ \zeta \end{pmatrix} : \zeta \in \mathbb{C}^2 \right\}, \tag{2.7}$$

where we use the 2×2 matrix representation with Pauli basis:

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

in $\mathrm{Mat}_{2\times 2}(\mathbb{C})$. This choice of η, ∞, $\mathbf{0}$ gives us the decomposition

$$\mathrm{su}(2,2) = \mathcal{T}_\infty \oplus \mathcal{L}_{0,\infty} \oplus \mathcal{D}_{0,\infty} \oplus \mathcal{A}_0, \tag{2.8}$$

where the subalgebras of four-translations, Lorentz, dilations and four-accelerations are given respectively by

$$\mathcal{T}_\infty = \left\{ \begin{pmatrix} 0 & T \\ 0 & 0 \end{pmatrix} : T = T^\dagger \in \mathrm{Mat}_{2\times 2}(\mathbb{C}) \text{ and } T = t^\mu \sigma_\mu \right\}, \tag{2.9a}$$

$$\mathcal{L}_{0,\infty} = \left\{ \begin{pmatrix} L & 0 \\ 0 & -L^\dagger \end{pmatrix} : \mathrm{Tr}\, L = 0 \text{ and } L \in \mathrm{Mat}_{2\times 2}(\mathbb{C}) \right\}, \tag{2.9b}$$

$$\mathcal{D}_{0,\infty} = \left\{ d \begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix} : d \in \mathbb{R} \right\}, \tag{2.9c}$$

$$\mathcal{A}_0 = \left\{ \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix} : C = C^\dagger \in \mathrm{Mat}_{2\times 2}(\mathbb{C}) \text{ and } C = c^\mu \sigma_\mu \right\}. \tag{2.9d}$$

The basis of $\mathrm{su}(2,2)^* \cong \mathrm{su}(2,2)$ dual to the one defined by Pauli matrices in the Lie subalgebras $\mathcal{T}_\infty, \mathcal{L}_{0,\infty}, \mathcal{D}_{0,\infty}, \mathcal{A}_0$ is

$$\mathcal{T}^*_\infty \ni \mathcal{P}^*_\mu = \begin{pmatrix} 0 & 0 \\ \sigma_\mu & 0 \end{pmatrix}, \tag{2.10a}$$

$$\mathcal{L}^*_{0,\infty} \ni \mathcal{L}^*_{kl} = \frac{1}{2}\epsilon_{klm} \begin{pmatrix} \sigma_m & 0 \\ 0 & \sigma_m \end{pmatrix}, \qquad \mathcal{L}^*_{0,\infty} \ni \mathcal{L}^*_{0k} = \frac{1}{2} \begin{pmatrix} \sigma_k & 0 \\ 0 & -\sigma_k \end{pmatrix}, \tag{2.10b}$$

$$\mathcal{D}^*_{0,\infty} \ni \mathcal{D}^* = \frac{1}{2} \begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix}, \tag{2.10c}$$

$$\mathcal{A}^*_0 \ni \mathcal{B}^*_\nu = \begin{pmatrix} 0 & \sigma_\nu \\ 0 & 0 \end{pmatrix}. \tag{2.10d}$$

One has the matrix coordinate map

$$\mathbb{M}_\infty \ni w \mapsto W \in \mathrm{Mat}_{2\times 2}(\mathbb{C}) \tag{2.11}$$

defined by

$$w = \left\{ \begin{pmatrix} W\zeta \\ \zeta \end{pmatrix} : \zeta \in \mathbb{C}^2 \right\} \tag{2.12}$$

and $w = x \in \mathbb{M}_\infty^{00}$ iff $W = W^\dagger = X$. The element $(x, \mathcal{X}) \in pr_1^{-1}(\mathbb{M}_\infty^{00})$ is parametrized by

$$(x, \mathcal{X}) \mapsto \left(X, \begin{bmatrix} XS & -XSX \\ S & -SX \end{bmatrix} \right), \tag{2.13}$$

where $X, S \in H(2)$ and $H(2)$ is the vector space of 2×2 Hermitian matrices.

The momentum maps (2.5) and (2.6) in the above defined coordinates are given by

$$J_0(X, S) = \begin{bmatrix} XS & -XSX \\ S & -SX \end{bmatrix}, \tag{2.14}$$

$$J_\lambda(W) = i\lambda \begin{bmatrix} (W + W^\dagger)(W - W^\dagger)^{-1} & -2W(W - W^\dagger)^{-1}W^\dagger \\ 2(W - W^\dagger)^{-1} & -\sigma_0 - 2(W - W^\dagger)^{-1}W^\dagger \end{bmatrix}. \tag{2.15}$$

By decomposing $J_0(X, S)$ in the basis (2.10)$J_0(X, S) = p_\mu \mathcal{P}_\mu^* + m_{\mu\nu}\mathcal{L}_{\mu\nu}^* + a_\mu \mathcal{A}_\nu^* + \mathrm{d}\mathcal{D}^*$ we obtain the expressions

$$p_\mu = s_\mu, \tag{2.16}$$

$$m_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu, \tag{2.17}$$

$$d = x^\mu p_\mu, \tag{2.18}$$

$$a_\mu = -2(x^\nu p_\nu)x_\mu + x^2 p_\mu \tag{2.19}$$

for the four-momentum p_μ, relativistic angular momentum $m_{\mu\nu}$, dilation d and four-acceleration a_ν respectively, where $S = s^\mu \sigma_\mu$, $X = x^\mu \sigma_\mu$.

In the coordinates x_μ, $p_\mu = s_\mu$ the symplectic form ω_0 assumes the canonical form

$$\omega_0 = \mathrm{d}x^\mu \wedge \mathrm{d}p_\mu. \tag{2.20}$$

Similarly, from $J_\lambda(W) = p_\mu \mathcal{P}_\mu^* + m_{\mu\nu}\mathcal{L}_{\mu\nu}^* + a_\mu \mathcal{A}_\nu^* + \mathrm{d}\mathcal{D}^*$ we obtain

$$p^\nu = \lambda \frac{y^\nu}{y^2}, \tag{2.21}$$

$$m_{\mu\nu} = x_\mu p_\nu - p_\nu x_\mu, \tag{2.22}$$

$$d = x^\mu p_\mu, \tag{2.23}$$

$$a_\mu = -2(x^\nu p_\nu)x_\mu + x^2 p_\mu - \frac{\lambda^2}{p^2} p_\mu, \tag{2.24}$$

where the real coordinates x_μ, y_μ on $\tilde{\mathbb{M}}$ are defined by $x^\nu + iy^\nu = w^\nu := \frac{1}{2}\mathrm{Tr}(W\sigma_\nu)$.

The coordinate description of ω_λ is the following

$$\omega_\lambda = i\lambda \frac{\partial^2}{\partial w^\mu \partial \bar{w}^\nu} \log(w - \bar{w})^2 \, \mathrm{d}w^\mu \wedge \mathrm{d}\bar{w}^\nu = \mathrm{d}x^\nu \wedge \mathrm{d}p_\nu. \tag{2.25}$$

Concluding, one has two models $(\mathbb{N}, \omega_0)$ and $(\mathbb{M}, \omega_\lambda)$ of the massive scalar conformal particle. Using the canonical coordinates (x^μ, p_ν) common for both models we obtain that

(i) the element $(x, \mathcal{X}) \in \mathbb{N}^{++}(w \in \mathbb{M}^{++})$ iff $p^0 > 0$ and $(p^0)^2 - \vec{p}^2 > 0$, i.e. it describes the state of a conformal scalar massive particle;
(ii) the element $(x, \mathcal{X}) \in \mathbb{N}^{--}(w \in \mathbb{M}^{--})$ iff $p^0 < 0$ and $(p^0)^2 - \vec{p}^2 > 0$, i.e. it describes the state of a conformal scalar massive anti-particle;
(iii) the element $(x, \mathcal{X}) \in \mathbb{N}^{+-}(w \in \mathbb{M}^{+-})$ iff $(p^0)^2 - \vec{p}^2 < 0$, i.e. it describes the state of a conformal scalar tachyon.

The orbits $\mathbb{N}^{0+}$ $(\mathbb{M}^{0+})$ and $\mathbb{N}^{0-}$ $(\mathbb{M}^{0-})$ describe the states of massless particles and anti-particles but this case will not be discussed further.

Two above presented models do not differ if one considers them on the level of relativistic mechanics, since both of them behave towards Poincaré transformations in the same way. The difference appears if one considers the four-acceleration transformations parametrized by $C = c^\mu \sigma_\mu$, which in canonical coordinates $X = x^\mu \sigma_\mu$, $P = p^\mu \sigma_\mu$ are

$$\tilde{X} = X(CX + \sigma_0)^{-1}, \tag{2.26}$$

$$\tilde{P} = (CX + \sigma_0)P(XC + \sigma_0) \tag{2.27}$$

for the standard model $\tilde{\mathbb{N}}$ and

$$\tilde{X} = [XP + i\lambda\sigma_0 - i\lambda(XC - i\lambda P^{-1}C + \sigma_0)](CXP + i\lambda C + P)^{-1}, \tag{2.28}$$

$$\tilde{P} = (CX + \sigma_0)P(XC + \sigma_0) + \lambda^2 C P^{-1} C \tag{2.29}$$

for the holomorphic model $\tilde{\mathbb{M}}$. We see from (2.29) that in the holomorphic model (opposite to the standard one) the four-momentum $P = p^\mu \sigma_\mu$ transforms in a non-linear way. This fact implies a lot of important physical consequences, e.g. the conformal scalar massive particle cannot be localized in the space-time in conformally invariant way. From (2.24), (2.28), (2.29) it follows that the holomorphic model corresponds to the nilpotent one when $\lambda \to 0$.

3. Conformally invariant quantum Kähler polarization

In this section we shall make the first step in the direction to construct quantum conformal phase space. Since the case of the antiparticle can be transformed by the charge conjugation map to the particle one, see [4], and the tachyon case is less interesting from physical point of view, we will work only with the phase space $\mathbb{M}^{++}$ of the conformal scalar massive particle.

The phase space $T^*\mathbb{M}^{00}$ has the real conformally invariant polarization defined by the leaves of its cotangent bundle structure. In canonical coordinates this polarization is spanned by the vector fields $\left\{\frac{\partial}{\partial p^\nu}\right\}_{\nu=0,\ldots,3}$. For the holomorphic phase space $\tilde{\mathbb{M}}$ the conformally invariant polarization is Kähler and in the complex coordinate it is spanned by $\left(\frac{\partial}{\partial \bar{w}^\mu}\right)_{\mu=0,\ldots,3}$. The reason is that

$SU(2,2)/\mathbb{Z}_4$ acts on $\tilde{\mathbb{M}}$ by biholomorphism. For $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SU(2,2)$ and $w \in \mathbb{M}^{++}$ one has

$$\sigma_g W = (AW + B)(CW + D)^{-1}, \tag{3.1}$$

where $W \in \mathrm{Mat}_{2\times 2}(\mathbb{C})$ is the matrix holomorphic coordinate of $w \in \mathbb{M}^{++}$. Using complex matrix coordinates (2.12) one identifies $\mathbb{M}^{++}$ with the future tube

$$\mathbb{T} := \{W \in \mathrm{Mat}_{2\times 2} : \mathrm{im}\, W > 0\}. \tag{3.2}$$

Applying the Caley transform

$$Z = (W - iE)(W + iE)^{-1}, \qquad W = i(Z + E)(Z - E)^{-1} \tag{3.3}$$

we map $\mathbb{T}$ on the symmetric domain

$$\mathbb{D} := \{Z \in \mathrm{Mat}_{2\times 2}(\mathbb{C}) : E - Z^* Z > 0\}. \tag{3.4}$$

Let us remark here that the coordinates $Z \in \mathbb{D}$ correspond to the diagonal representation of the twistor form $\eta = \begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix}$. Below we use both systems of coordinates.

In order to quantize $\mathbb{M}^{++}$ we will use the method of coherent state map investigated in [10]. For the other construction of noncommutative manifolds by using coherent state method see also [11]. The essence of this method consists in replacing the classical state $m \in \mathbb{M}^{++}$ by the quantum pure state, which means, that one defines the map $\mathcal{K}_\lambda : \mathbb{M}^{++} \mapsto \mathbb{CP}(\mathcal{H})$ from the classical phase space $\mathbb{M}^{++}$ into the complex projective separable Hilbert space $\mathbb{CP}(\mathcal{H})$. We will call $\mathcal{K}_\lambda$ coherent state map and in our case we will postulate that it has the following properties:

(i) $\mathcal{K}_\lambda$ is consistent with the conformal symmetry, i.e. there exists an unitary irreducible representation $\mathbf{U}_\lambda : SU(2,2) \mapsto \mathrm{Aut}\,\mathcal{H}$ with respect to which the coherent state map is equivariant:

$$\begin{tikzcd} \mathbb{M}^{++} \arrow[r, "\mathcal{K}_\lambda"] \arrow[d, "\sigma_g"'] & \mathbb{CP}(\mathcal{H}) \arrow[d, "{[\mathbf{U}_\lambda(g)]\ \forall g \in SU(2,2)}"] \\ \mathbb{M}^{++} \arrow[r, "\mathcal{K}_\lambda"'] & \mathbb{CP}(\mathcal{H}) \end{tikzcd} \tag{3.5}$$

(ii) $\mathcal{K}_\lambda$ is consistent with the holomorphic polarization $\left(\frac{\partial}{\partial \bar{w}^\mu}\,_{\mu=0,\ldots,3}\right)$. This denotes that $\mathcal{K}_\lambda$ is a holomorphic map.

(iii) $\mathcal{K}_\lambda$ is symplectic, i.e.

$$\mathcal{K}_\lambda^* \omega_{FS} = \omega_\lambda, \tag{3.6}$$

where ω_{FS} is Fubini-Study form on $\mathbb{CP}(\mathcal{H})$. The projective Hilbert space is considered here as Kähler manifold (thus symplectic manifold). This condition one needs for the consistence of classical dynamics with quantum dynamics.

The coherent state map $\mathcal{K}_\lambda : \mathbb{M}^{++} \mapsto \mathbb{CP}(\mathcal{H})$ fulfilling the properties postulated above one obtains by the applying of the representation theory, see [12,13]. We skip here the technical considerations and present only the final result. Let

$$\left\{ \left| \begin{matrix} j & m \\ j_1 & j_2 \end{matrix} \right\rangle \right\}, \tag{3.7}$$

where $m, 2j \in \mathbb{N} \cup \{0\}$ and $-j \leq j_1,\, j_2 \leq j$, denote an orthonormal basis in $\mathcal{H}$, i.e.

$$\left\langle \begin{matrix} j & m \\ j_1 & j_2 \end{matrix} \middle| \begin{matrix} j' & m' \\ j'_1 & j'_2 \end{matrix} \right\rangle = \delta_{jj'}\delta_{mm'}\delta_{j_1 j'_1}\delta_{j_2 j'_2}. \tag{3.8}$$

Then the map $K_\lambda : \mathbb{M}^{++} \cong \mathbb{D} \mapsto \mathcal{H}$ given by

$$K_\lambda : Z \to |Z;\lambda\rangle := \sum_{j,m,j_1,j_2} \Delta^{jm}_{j_1 j_2}(Z) \left| \begin{matrix} j & m \\ j_1 & j_2 \end{matrix} \right\rangle, \tag{3.9}$$

where

$$\begin{aligned} \Delta^{jm}_{j_1 j_2}(Z) :&= (N^\lambda_{jm})^{-1} (\det\, Z)^m \sqrt{\frac{(j+j_1)!(j-j_1)!}{(j+j_2)!(j-j_2)!}} \\ &\quad \times \sum_{\substack{S \geq \max\{0, j_1+j_2\} \\ S \leq \min\{j+j_1, j+j_2\}}} \begin{pmatrix} j+j_2 \\ S \end{pmatrix} \begin{pmatrix} j-j_2 \\ S-j_1-j_2 \end{pmatrix} \\ &\quad \times z_{11}^S z_{12}^{j+j_1-S} z_{21}^{j+j_2-S} z_{22}^{S-j_1-j_2} \end{aligned} \tag{3.10}$$

and

$$N^\lambda_{jm} := (\lambda-1)(\lambda-2)^2(\lambda-3) \frac{\Gamma(\lambda-2)\Gamma(\lambda-3) m!(m+2j+1)!}{(2j+1)!\Gamma(m+\lambda-1)\Gamma(m+2j+\lambda)}, \tag{3.11}$$

defines a coherent state map

$$[K_\lambda] =: \mathcal{K}_\lambda : \mathbb{M}^{++} \mapsto \mathbb{CP}(\mathcal{H}) \tag{3.12}$$

with the properties mentioned in assumptions: (i), (ii), (iii). The condition (i) restricts the variability of the parameter $\lambda > 3$ to integer numbers.

From now on, to simplify the notation, we will write $|Z\rangle$ instead of $|Z;\lambda\rangle$. If the dependence on λ is relevant we will write $|Z;\lambda\rangle$.

The projectors $\frac{|Z\rangle\langle Z|}{\langle Z|Z\rangle}$ representing the coherent states give the resolution of the identity

$$\mathbf{1} = \int_{\mathbb{D}} |Z\rangle\langle Z| \,\mathrm{d}\mu_\lambda(Z, Z^\dagger) \tag{3.13}$$

with respect to the measure

$$\mathrm{d}\mu_\lambda(Z, Z^\dagger) = c_\lambda [\det(E - Z^\dagger Z)]^{\lambda-4} |\,\mathrm{d}Z|, \tag{3.14}$$

where $|\mathrm{d}Z|$ is the Lebesgue measure on $\mathbb{D}$ and

$$c_\lambda = \pi^{-4}(\lambda-1)(\lambda-2)^2(\lambda-3), \tag{3.15}$$

which is equivalent to $\int_{\mathbb{D}} \mathrm{d}\mu_\lambda = 1$.

Hence, by the anti-linear monomorphism

$$I_\lambda : \mathcal{H} \ni |\psi\rangle \to \langle\psi|\cdot;\lambda\rangle := \psi(\cdot) \in \mathcal{O}(\mathbb{D}) \tag{3.16}$$

one identifies $\mathcal{H}$ with the range of I_λ in $\mathcal{O}(\mathbb{D})$, which is equal to the Hilbert space of holomorphic functions $L^2\mathcal{O}(\mathbb{D}, \mathrm{d}\mu_\lambda)$ square integrable with respect to the measure (3.14).

The representation $I_\lambda \circ U_\lambda \circ I_\lambda^{-1}$ acts on $L^2\mathcal{O}(\mathbb{D}, \mathrm{d}\mu_\lambda)$ by

$$(I_\lambda \circ U_\lambda(g) \circ I_\lambda^{-1})\psi(Z) = [\det(\sim C\, Z + \sim D)]^{-\lambda}\psi(\sigma_{g-1}(Z)), \tag{3.17}$$

where $g^{-1} = \begin{pmatrix} \sim A & \sim B \\ \sim C & \sim D \end{pmatrix}$ i.e. it is a discrete series representation of SU(2, 2) and acts on the coherent states by

$$U_\lambda(g)|Z\rangle = [\det(CZ + D)]^{-\lambda}|\sigma_g(Z)\rangle, \tag{3.18}$$

where $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{SU}(2, 2)$, see [14,15].

The 15 physical quantities p_ν, $m_{\mu\nu}$, d and a_ν, μ, $\nu = 0, 1, 2, 3$, which characterize the scalar massive conformal particle form the conformal Lie algebra su(2, 2) with respect to the Poisson bracket

$$\{f, g\}_\lambda(\bar{w}, w) = \frac{i}{2\lambda}((w - \bar{w})^2\eta^{\mu\nu} - 2(w^\mu - \bar{w}^\mu)(w^\nu - \bar{w}^\nu)) \times \left(\frac{\partial f}{\partial w^\mu}\frac{\partial g}{\partial \bar{w}^\nu} - \frac{\partial g}{\partial w^\mu}\frac{\partial f}{\partial \bar{w}^\nu}\right) \tag{3.19}$$

defined by the symplectic form ω_λ. Each one of them defines a Hamiltonian flow $\sigma_{g(t)}$ on $\mathbb{M}^{++}$ realized by the corresponding one-parameter subgroup $g(t)$, $t \in \mathbb{R}$, of SU(2, 2). By the equivariance condition (4.35) this Hamiltonian flow $\sigma_{g(t)}$ is quantized to the Hamiltonian flow on $\mathbb{CP}(\mathcal{H})$ given by the one-parameter subgroup $\mathbf{U}_\lambda(g(t))$ of representation (3.18). The generators of these one-parameter subgroups are realized in $L^2\mathcal{O}(\mathbb{T}, \mathrm{d}\mu_\lambda)$ as follows:

$$\hat{p_\mu} = -i\frac{\partial}{\partial w^\mu}, \tag{3.20}$$

$$\hat{m_{\mu\nu}} = -i\left(w_\mu\frac{\partial}{\partial w^\nu} - w_\nu\frac{\partial}{\partial w^\mu}\right), \tag{3.21}$$

$$\hat{d} = -2iw^\mu\frac{\partial}{\partial w^\mu} - 2i\lambda, \tag{3.22}$$

$$\hat{a_\nu} = -iw^2(\delta_\nu^\beta - 2w_\nu w^\beta)\frac{\partial}{\partial w^\beta} + 2i\lambda w_\nu, \tag{3.23}$$

see [6]. They are quantized versions of their classical counterparts given by (2.16)–(2.19). The measure $\mathrm{d}\mu_\lambda$ in the future tube representation is given by

$$\mathrm{d}\mu_\lambda(W, W^\dagger) = 2^{-4}[\det(W - W^\dagger)]^{\lambda-4}|\,\mathrm{d}W|. \tag{3.24}$$

It was shown in [10] that the coherent state method of quantization is equivalent to the Kostant–Souriou geometric quantization.

Besides generators (2.16)–(2.19) of the conformal Lie algebra su(2, 2) there is also reason to quantize other physically important observables. In particular case the ones belonging to the family $\mathcal{O}^{++}(\mathbb{D})$ consisting of complex valued smooth functions $f : \mathbb{M}^{++} \to \mathbb{C}$ for whose there exists bounded operators $a(f) \in L^\infty(\mathcal{H})$ such that

$$a(f)|Z\rangle = f(Z)|Z\rangle \tag{3.25}$$

for any $Z \in \mathbb{D} \cong \mathbb{M}^{++}$. Since the coherent states $|Z\rangle$ form a linearly dense subset of $\mathcal{H}$ one has correctly defined linear map $a : \mathcal{O}^{++}(\mathbb{D}) \to L^\infty(\mathcal{H})$ of $\mathcal{O}^{++}(\mathbb{D})$ in the Banach algebra of the bounded operators.

It follows immediately from (3.25) and the resolution of identity (3.13) that

(i) $\mathcal{O}^{++}(\mathbb{D})$ is the commutative algebra and $f \in \mathcal{O}^{++}(\mathbb{D})$ is holomorphic.
(ii) The map $a : \mathcal{O}^{++}(\mathbb{D}) \to L^\infty(\mathcal{H})$ is an isometric

$$\|a(f)\|_\infty = \|f\|_{\sup} = \sup_{Z\in\mathbb{D}} |f(Z)| \tag{3.26}$$

monomorphism of algebras.
(iii) The image $a(\mathcal{O}^{++}(\mathbb{D}))$ is uniformly closed in $L^\infty(\mathcal{H})$ (i.e. with respect to operator norm $\|\cdot\|_\infty$).

Hence, $\mathcal{O}^{++}(\mathbb{D})$ is a Banach subalgebra of the Banach algebra $H^\infty(\mathbb{D})$ of functions which are holomorphic and bounded on $\mathbb{D}$. Let us remark here that completeness of $H^\infty(\mathbb{D})$ follows from the Weierstrass theorem, see e.g. [16].

Indeed one has the following proposition.

Proposition 1. *The Banach algebra* $\mathcal{O}^{++}(\mathbb{D})$ *is equal to* $H^\infty(\mathbb{D})$.

Proof. Since $I_\lambda(\mathcal{H}) = L^2\mathcal{O}(\mathbb{D}, d\mu_\lambda)$ we have $f\langle\psi|\cdot\rangle \in I_\lambda(\mathcal{H})$ for any $f \in H^\infty(\mathbb{D})$. The multiplication operator $M_f : L^2\mathcal{O}(\mathbb{D}, d\mu_\lambda) \to L^2\mathcal{O}(\mathbb{D}, d\mu_\lambda)$ is bounded. Thus there is a bounded operator $a(f)^* : \mathcal{H} \to \mathcal{H}$ such that

$$f(Z)\langle\psi|Z\rangle = \langle a(f)^*\psi|Z\rangle \tag{3.27}$$

for $Z \in \mathbb{D}$. The above shows that $a(f) = (a(f)^*)^*$ fulfills (3.25). □

According to [17] we shall call the commutative Banach algebra $\mathcal{P}^{++} := a(H^\infty(\mathbb{D}))$ the *quantum Kähler polarization* and its elements $a(f) \in \mathcal{P}^{++}$ the *annihilation operators*.

The coordinate functions $f_{kl}(Z) := z_{kl}$, where $k, l = 1, 2$ belong to $H^\infty(\mathbb{D})$. Therefore $a_{kl} := a(f_{kl}) \in \mathcal{P}^{++}$ and their action on the basis (3.7) is given by

$$a_{11}\left|\begin{matrix} j & m \\ j_1 & j_2 \end{matrix}\right\rangle = \sqrt{\frac{(j-j_1+1)(j-j_2+1)m}{(2j+1)(2j+2)(m+\lambda-2)}}\left|\begin{matrix} j+\frac{1}{2} & m-1 \\ j_1-\frac{1}{2} & j_2-\frac{1}{2} \end{matrix}\right\rangle + \sqrt{\frac{(j+j_1)(j+j_2)(m+2j+1)}{(m+2j+\lambda-1)2j(2j+1)}}\left|\begin{matrix} j-\frac{1}{2} & m \\ j_1-\frac{1}{2} & j_2-\frac{1}{2} \end{matrix}\right\rangle, \tag{3.28}$$

$$a_{12}\left|\begin{array}{cc} j & m \\ j_1 & j_2 \end{array}\right\rangle = -\sqrt{\frac{(j-j_1+1)(j+j_2+1)m}{(2j+1)(2j+2)(m+\lambda-2)}}\left|\begin{array}{cc} j+\frac{1}{2} & m-1 \\ j_1-\frac{1}{2} & j_2+\frac{1}{2} \end{array}\right\rangle$$
$$+\sqrt{\frac{(j+j_1)(j-j_2)(m+2j+1)}{(m+2j+\lambda-1)2j(2j+1)}}\left|\begin{array}{cc} j-\frac{1}{2} & m \\ j_1-\frac{1}{2} & j_2+\frac{1}{2} \end{array}\right\rangle, \quad (3.29)$$

$$a_{21}\left|\begin{array}{cc} j & m \\ j_1 & j_2 \end{array}\right\rangle = -\sqrt{\frac{(j+j_1+1)(j-j_2+1)m}{(2j+1)(2j+2)(m+\lambda-2)}}\left|\begin{array}{cc} j+\frac{1}{2} & m-1 \\ j_1+\frac{1}{2} & j_2-\frac{1}{2} \end{array}\right\rangle$$
$$+\sqrt{\frac{(j-j_1)(j+j_2)(m+2j+1)}{(m+2j+\lambda-1)2j(2j+1)}}\left|\begin{array}{cc} j-\frac{1}{2} & m \\ j_1+\frac{1}{2} & j_2-\frac{1}{2} \end{array}\right\rangle, \quad (3.30)$$

$$a_{22}\left|\begin{array}{cc} j & m \\ j_1 & j_2 \end{array}\right\rangle = \sqrt{\frac{(j+j_1+1)(j+j_2+1)m}{(2j+1)(2j+2)(m+\lambda-2)}}\left|\begin{array}{cc} j+\frac{1}{2} & m-1 \\ j_1+\frac{1}{2} & j_2+\frac{1}{2} \end{array}\right\rangle$$
$$+\sqrt{\frac{(j-j_1)(j-j_2)(m+2j+1)}{(m+2j+\lambda-1)2j(2j+1)}}\left|\begin{array}{cc} j-\frac{1}{2} & m \\ j_1+\frac{1}{2} & j_2+\frac{1}{2} \end{array}\right\rangle. \quad (3.31)$$

In the expressions above we put by definition

$$\left|\begin{array}{cc} j & m \\ j_1 & j_2 \end{array}\right\rangle := 0$$

if the indices do not satisfy the condition $m, 2j \in \mathbb{N}\cup\{0\}$ and $-j \leq j_1, j_2 \leq j$.

The coordinate annihilation operators a_{kl}, $k, l = 1, 2$ generate Banach subalgebra $\mathcal{P}^{++}_{\mathrm{pol}}$ of $\mathcal{P}^{++}$. Let us denote by $\mathrm{Pol}(\bar{\mathbb{D}})$ the algebra of polynomials of variables $\{z_{kl}\}$, $k, l = 1, 2$ restricted to the closure $\bar{\mathbb{D}}$ of $\mathbb{D}$ in $\mathrm{Mat}_{2\times 2}(\mathbb{C})$.

For the following considerations let us fix the matrix notation

$$\mathbb{A} := \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathcal{P}^{++}_{\mathrm{pol}} \otimes \mathrm{Mat}_{2\times 2}(\mathbb{C}), \quad (3.32)$$

$$\mathbb{A}^+ := \begin{pmatrix} a^*_{11} & a^*_{21} \\ a^*_{12} & a^*_{22} \end{pmatrix} \in \overline{\mathcal{P}^{++}_{\mathrm{pol}}} \otimes \mathrm{Mat}_{2\times 2}(\mathbb{C}) \quad (3.33)$$

for the annihilation and creation operators. For example, in this notation the property (3.25) assumes the form

$$\mathbb{A}|Z\rangle = Z|Z\rangle. \tag{3.34}$$

Proposition 2.

(i) $\mathcal{P}^{++}_{\text{pol}}$ *is isometrically isomorphic to the closure* $\overline{Pol(\bar{\mathbb{D}})}$ *of* $Pol(\bar{\mathbb{D}})$, *i.e.* $a(f) \in \mathcal{P}^{++}_{\text{pol}}$ *iff* f *is continuous on* $\bar{\mathbb{D}}$ *and holomorphic on* $\mathbb{D}$. *The space of maximal ideals of the* $\mathcal{P}^{++}_{\text{pol}}$ (*the spectrum*) *is homeomorphic to* $\bar{\mathbb{D}}$.
(ii) $\mathcal{P}^{++}_{\text{pol}}$ *is a semisimple Banach algebra, i.e. if* $p \in \mathcal{P}^{++}_{\text{pol}}$ *is such that for each* $c \in \mathbb{C}$ *there exists* $(1+cp)^{-1}$ *then* $p = 0$.
(iii) $\mathcal{P}^{++}_{\text{pol}} \subsetneq \mathcal{P}^{++}$, *i.e. it is proper Banach subalgebra of* $\mathcal{P}^{++}$.
(iv) *The vacuum state is cyclic with respect to the Banach algebra* $\overline{\mathcal{P}^{++}_{\text{pol}}}$.

Proof.

(i) For $Z, W \in \bar{\mathbb{D}}$ and $\alpha \in [0, 1]$ one has

$$\begin{aligned} &v^\dagger(E - [\alpha Z + (1-\alpha)W]^\dagger[\alpha Z + (1-\alpha)W])v \\ &\quad = \|v\|^2 - \|[\alpha Z + (1-\alpha)W]v\|^2 \geq \|v\|^2 - \{\alpha\|Zv\| + (1-\alpha)\|Wv\|\}^2 \\ &\qquad \geq \|v\|^2 - \{\alpha\|v\| + (1-\alpha)\|v\|\}^2 = 0, \end{aligned} \tag{3.35}$$

for each $v \in \mathbb{C}^2$, what gives $\alpha Z + (1-\alpha)W \in \bar{\mathbb{D}}$. So, $\bar{\mathbb{D}}$ is convex bounded subset of $\mathrm{Mat}_{2\times 2}(\mathbb{C})$. Thus $\bar{\mathbb{D}}$ is polynomially convex and compact. By definition $\mathcal{P}^{++}_{\text{pol}}$ has a finite number of generators. Hence statement (i) is valid, see for example Chapter 7 of [18].
(ii) We recall that the radical of algebra $\mathcal{P}^{++}$ is

$$\mathcal{R} = \{b \in \mathcal{P}^{++} : (b + \lambda\mathbb{I}) \text{ is invertible for any } \lambda \neq 0\}. \tag{3.36}$$

Statement (ii) is valid since every $b \in \mathcal{P}^{++}$ has non-zero eigenvalue.
(iii) To prove this it is enough to find a function $f \in H^\infty(\mathbb{D})$ such that $f \notin \overline{Pol(\bar{\mathbb{D}})}$. For example the function

$$f(Z) = \exp\frac{\mathrm{Tr}(Z+E)}{\mathrm{Tr}(Z-E)} \tag{3.37}$$

has this property.
(iv) It is enough to check that

$$\left|\begin{matrix} j & m \\ j_1 & j_2 \end{matrix}\right\rangle = \Delta^{jm}_{j_1 j_2}(\mathbb{A}^\dagger)\left|\begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix}\right\rangle \tag{3.38}$$

and notice that operator $\Delta^{jm}_{j_1 j_2}(\mathbb{A}^\dagger) \in \overline{\mathcal{P}^{++}_{\text{pol}}}$. □

We define the action of the $g \in \mathrm{SU}(2, 2)$ on $\mathbb{A}$ by

$$\mathbf{U}_\lambda(g)\mathbb{A}\mathbf{U}_\lambda(g^{-1}) := \begin{pmatrix} U_\lambda(g)a_{11}U_\lambda(g^{-1}) & U_\lambda(g)a_{12}U_\lambda(g^{-1}) \\ U_\lambda(g)a_{21}U_\lambda(g^{-1}) & U_\lambda(g)a_{22}U_\lambda(g^{-1}) \end{pmatrix}, \tag{3.39}$$

where $\mathrm{SU}(2,2) \ni g \to U_\lambda(g) \in \mathrm{Aut}(\mathcal{H})$ is discrete series representation defined by (3.17). Using the above notation we formulate the following statement.

Proposition 3. *One has*

(i) $$\sigma_g(\mathbb{A}) := (A\mathbb{A} + B)(C\mathbb{A} + D)^{-1} \in \mathcal{P}^{++}_{\mathrm{pol}} \otimes Mat_{2\times 2}(\mathbb{C}), \tag{3.40}$$

(ii) $$\mathbf{U}_\lambda(g^{-1})\mathbb{A}\mathbf{U}_\lambda(g) = \sigma_g(\mathbb{A}), \tag{3.41}$$

for $g \in \mathrm{SU}(2,2)$.

Proof.

(i) For

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{SU}(2,2)$$

one has

$$DD^\dagger = E + CC^\dagger. \tag{3.42}$$

So eigenvalues of $DD^\dagger$ satisfy $d_1, d_2 \geq 1$, which implies that

$$\begin{aligned} \|D^{-1}CZ\|^2 \leq \|D^{-1}C(D^{-1}C)^\dagger\| &= \|D^{-1}(DD^\dagger - E)D^{\dagger^{-1}}\| \\ &= \|E - D^{-1}D^{\dagger^{-1}}\| < 1 \end{aligned} \tag{3.43}$$

for $Z \in \bar{\mathbb{D}}$. The above gives that the inverse $(D + CZ)^{-1} = (E + D^{-1}CZ)^{-1}D^{-1}$ exists for $Z \in \bar{\mathbb{D}}$. Since $\det(CZ + D)$ is continuous function of Z and $\det(CZ + D) \neq 0$ for $z \in \bar{\mathbb{D}}$ there exists $\Omega \supset \bar{\mathbb{D}}$ such that $\det(CZ + D) \neq 0$ for all $z \in \Omega$. This shows that the matrix function

$$\sigma_g(\mathbb{Z}) = (A\mathbb{Z} + B)(C\mathbb{Z} + D)^{-1} \tag{3.44}$$

is holomorphic on Ω. So, by the Oka–Weil theorem, see [18,19], there is a sequence $\{p_n\}$ of polynomials in $z_{11}, z_{12}, z_{21}, z_{22}$ with $p_n \to \sigma_g$ uniformly on $\bar{\mathbb{D}}$. Since $\mathcal{P}^{++}_{\mathrm{pol}} \cong \overline{\mathrm{Pol}(\bar{\mathbb{D}})}$ one proves $\sigma_g(\mathbb{A}) \in \mathcal{P}^{++}_{\mathrm{pol}} \otimes \mathrm{Mat}_{2\times 2}(\mathbb{C})$.

(ii) Let us note that for a linearly dense set of the coherent states $|Z\rangle$, $Z \in \mathbb{D}$,

$$\mathbf{U}_\lambda(g^{-1})\mathbb{A}\mathbf{U}_\lambda(g)|Z\rangle = \sigma_g(\mathbb{A})|Z\rangle \tag{3.45}$$

which gives (3.41). □

We conclude immediately from Proposition 3.

Corollary 4. *Banach subalgebra* $\mathcal{P}^{++}_{\mathrm{pol}} \subset L^\infty(\mathcal{H})$ *is invariant* $U_\lambda(g)\mathcal{P}^{++}_{\mathrm{pol}}U_\lambda(g^{-1}) \subset \mathcal{P}^{++}_{\mathrm{pol}}$, $g \in \mathrm{SU}(2,2)$ *with respect to the discrete series representation*.

Let us make a closing remark that quantum polarization $\mathcal{P}^{++}$ gives holomorphic operator coordinatization for the classical phase space $\mathbb{M}^{++}$ and subalgebra $\mathcal{P}^{++}_{\mathrm{pol}} \subset \mathcal{P}^{++}$ gives the coordinatization of $\mathbb{M}^{++}$ algebraic in the annihilation operators.

4. Conformal Kähler quantum phase space

The holomorphic quantum coordinatization of the classical phase space $\mathbb{M}^{++}$ by the operator Banach algebra $\mathcal{P}^{++}$ is not sufficient from the physical point of view. The reason is that the complete quantum description of the scalar conformal particle also requires self-adjoint operators, for example such as those given by (3.20)–(3.23). Therefore, we are obliged to include in our considerations the Banach algebra $\overline{\mathcal{P}^{++}}$ generated by the creation operators a^*_{kl}, $k, l = 1, 2$, which by definition are conjugated counterparts of the annihilation operators. The algebra $\overline{\mathcal{P}^{++}}$ gives anti-holomorphic quantum coordinatization of $\mathbb{M}^{++}$. From Proposition 3 it follows that $\overline{\mathcal{P}^{++}}$ as well as $\mathcal{P}^{++}$ are conformally invariant *quantum Kähler polarizations* on $\mathbb{M}^{++}$. Then, following [17], we shall call the operator C^*-algebra $\mathcal{M}^{++} \subset L^{\infty}(\mathcal{H})$ generated by $\mathcal{P}^{++}$ the *quantum Kähler phase space* of the scalar conformal particle. We shall denote by $\mathcal{M}^{++}_{\text{pol}}$ the proper C^*-subalgebra of $\mathcal{M}^{++}$ generated by $\mathcal{P}^{++}_{\text{pol}} \subsetneq \mathcal{P}^{++}$.

The relation between the quantum phase space $\mathcal{M}^{++}$ and its classical mechanical counterpart $\mathbb{M}^{++}$ is best seen by the covariant and contravariant symbols description.

For any bounded operator $F \in L^{\infty}(\mathcal{H})$ one defines the *two-covariant symbol*

$$\langle F\rangle_2(Z^{\dagger}, V) := \frac{\langle Z|FV\rangle}{\langle Z|V\rangle}. \tag{4.1}$$

The *two-contravariant symbol* f is defined as an element of the space $\mathcal{B}_2(\mathbb{D}\times\mathbb{D})$ of complex valued functions on $\mathbb{D}\times\mathbb{D}$ for which the integral

$$F = \mathcal{F}_{\lambda}(f) := c_{\lambda}^2 \int_{\mathbb{D}\times\mathbb{D}} f(Z^{\dagger}, V)\frac{|Z\rangle\langle Z|V\rangle\langle V|}{\langle Z|Z\rangle\langle V|V\rangle}\,\mathrm{d}\mu(Z^{\dagger}, Z)\,\mathrm{d}\mu(V^{\dagger}, V) \tag{4.2}$$

exists weakly and $\mathcal{F}_{\lambda}(f) \in L^{\infty}(\mathcal{H})$, where the measure $\mathrm{d}\mu$ is defined by

$$\mathrm{d}\mu(Z^{\dagger}, Z) = \det(E - Z^{\dagger}Z)^{-4}\,|\mathrm{d}z|\,. \tag{4.3}$$

We define:

(i) the associative product

$$\begin{aligned}(f \bullet_{\lambda} g)(Z^{\dagger}, W) :&= = c_{\lambda}^2 \int_{\mathbb{D}\times\mathbb{D}} f(Z^{\dagger}, V)g(S^{\dagger}, W)\\ &\quad\times \frac{\langle Z|V\rangle\langle V|S\rangle\langle S|W\rangle}{\langle Z|W\rangle\langle V|V\rangle\langle S|S\rangle}\,\mathrm{d}\mu(V^{\dagger}, V)\,\mathrm{d}\mu(S^{\dagger}, S)\\ &= \int_{\mathbb{D}\times\mathbb{D}} f(Z^{\dagger}, V)g(S^{\dagger}, W)\frac{\langle Z|V\rangle\langle V|S\rangle\langle S|W\rangle}{\langle Z|W\rangle}\\ &\quad\times \mathrm{d}\mu_{\lambda}(V^{\dagger}, V)\,\mathrm{d}\mu_{\lambda}(S^{\dagger}, S),\end{aligned} \tag{4.4}$$

of the two-contravariant symbols $f, g \in \mathcal{B}_2(\mathbb{D}\times\mathbb{D})$;

(ii) the seminorm

$$\|f\| := \|\mathcal{F}_{\lambda}(f)\|_{\infty} \tag{4.5}$$

and the involution

$$f^*(Z^{\dagger}, V) := \overline{f(V, Z^{\dagger})} \tag{4.6}$$

of the two-contravariant symbol. The map $\mathcal{F}_\lambda : \mathcal{B}_2(\mathbb{D} \times \mathbb{D}) \to L^\infty(\mathcal{H})$ is an epimorphism of algebras with involution and

$$\ker \mathcal{F}_\lambda = \{f \in \mathcal{B}_2(\mathbb{D} \times \mathbb{D}) : \|f\| = 0\}. \tag{4.7}$$

Thus the quotient algebra $\mathcal{B}_2(\mathbb{D} \times \mathbb{D})/\ker \mathcal{F}_\lambda$ and $L^\infty(\mathcal{H})$ are isomorphic as C^*-algebras. Since each equivalence class $[f] = f + \ker \mathcal{F}_\lambda$ is represented in a unique way by the two-covariant symbol $\langle \mathcal{F}_\lambda(f)\rangle_2$, i.e. $[f] = \langle \mathcal{F}_\lambda(f)\rangle_2 + \ker \mathcal{F}_\lambda$ and $\langle \mathcal{F}_\lambda(f)\rangle_2 = \langle \mathcal{F}_\lambda(g)\rangle_2$ iff $f - g \in \ker \mathcal{F}_\lambda$, then the quotient vector space $\mathcal{B}_2(\mathbb{D} \times \mathbb{D})/\ker \mathcal{F}_\lambda$ is isomorphic with the vector space

$$\mathcal{B}^2(\mathbb{D} \times \mathbb{D}) := \{\langle F\rangle_2 : F \in L^\infty(\mathcal{H})\} \tag{4.8}$$

of two-covariant symbols of the bounded operators. Defining the product of the two-covariant symbols $\langle F\rangle_2, \langle G\rangle_2 \in \mathcal{B}^2(\mathbb{D} \times \mathbb{D})$ by

$$\langle F\rangle_2 *_\lambda \langle G\rangle_2(Z^\dagger, V) := c_\lambda \int \langle F\rangle_2(Z^\dagger, W)\langle G\rangle_2(W^\dagger, V)\frac{\langle Z|W\rangle\langle W|V\rangle}{\langle W|W\rangle\langle Z|V\rangle}\,\mathrm{d}\mu(W^\dagger, W) \tag{4.9}$$

one obtains the structure of C^*-algebra on $\mathcal{B}^2(\mathbb{D} \times \mathbb{D})$.

The quotient map $\mathcal{B}_2(\mathbb{D} \times \mathbb{D}) \to \mathcal{B}_2(\mathbb{D} \times \mathbb{D})/\ker \mathcal{F}_\lambda$ and the isomorphism $\mathcal{B}_2(\mathbb{D} \times \mathbb{D})/\ker \mathcal{F}_\lambda \cong \mathcal{B}^2(\mathbb{D} \times \mathbb{D})$ defines the epimorphism

$$\pi : \mathcal{B}_2(\mathbb{D} \times \mathbb{D}) \longrightarrow \mathcal{B}^2(\mathbb{D} \times \mathbb{D}) \tag{4.10}$$

of the algebra with involution $(\mathcal{B}_2(\mathbb{D} \times \mathbb{D}), \bullet_\lambda)$ on the C^*-algebra $(\mathcal{B}^2(\mathbb{D} \times \mathbb{D}), *_\lambda)$. Similarly the inclusion map

$$\iota : \mathcal{B}^2(\mathbb{D} \times \mathbb{D}) \hookrightarrow \mathcal{B}_2(\mathbb{D} \times \mathbb{D}) \tag{4.11}$$

is the monomorphism of C^*-algebra $(\mathcal{B}^2(\mathbb{D} \times \mathbb{D}), *_\lambda)$ to the algebra $(\mathcal{B}_2(\mathbb{D} \times \mathbb{D}), \bullet_\lambda)$.

In the case under consideration the coherent state map $\mathcal{K}_\lambda : \mathbb{D} \to \mathbb{CP}(\mathcal{H})$ is holomorphic and $\mathbb{D}$ is a simply connected domain. Hence one can reconstruct the two-covariant symbol $\langle F\rangle_2$ of the bounded operator $F \in L^\infty(\mathcal{H})$ from its *Berezin covariant symbol*

$$\langle F\rangle(Z^\dagger, Z) := \frac{\langle Z|FZ\rangle}{\langle Z|Z\rangle}. \tag{4.12}$$

The reconstruction is given by the analytic continuation of $\langle F\rangle$ from the diagonal $\delta : \mathbb{D} \cong \varDelta \hookrightarrow \mathbb{D} \times \mathbb{D}$ to the product $\mathbb{D} \times \mathbb{D}$. As a result we obtain the linear isomorphism

$$c : \mathcal{B}(\mathbb{D}) \xrightarrow{\sim} \mathcal{B}^2(\mathbb{D} \times \mathbb{D}) \tag{4.13}$$

of the vector space $\mathcal{B}(\mathbb{D}) := \{\langle F\rangle : F \in L^\infty(\mathcal{H})\}$ of Berezin covariant symbols with $\mathcal{B}^2(\mathbb{D} \times \mathbb{D})$. The map (4.13) is inverse to the restriction map

$$\delta^* : \mathcal{B}^2(\mathbb{D} \times \mathbb{D}) \ni \langle F\rangle_2 \longrightarrow \langle F\rangle_2 \circ \delta \in \mathcal{B}(\mathbb{D}). \tag{4.14}$$

Hence one also defines the product

$$f *_\lambda g := \delta^*(c(f) *_\lambda c(g)) \tag{4.15}$$

of $f, g \in \mathcal{B}(\mathbb{D})$, which is given explicitly by

$$(f *_\lambda g)(Z^\dagger, Z) = c_\lambda \int_{\mathbb{D}} f(Z^\dagger, V)g(V^\dagger, Z)\left|a_\lambda(Z^\dagger, V)\right|^2 \mathrm{d}\mu(V^\dagger, V), \tag{4.16}$$

where

$$a_\lambda(Z^\dagger, V) := \frac{\langle Z|V\rangle}{\sqrt{\langle Z|Z\rangle\langle V|V\rangle}} \tag{4.17}$$

is the transition amplitude between the coherent states $\mathcal{K}_\lambda(Z)$ and $\mathcal{K}_\lambda(V)$. For brevity, by f and g in (4.15) we denoted the Berezin covariant symbols of $F, G \in L^\infty(\mathcal{H})$.

Let us visualize the morphisms defined above in the following diagram

$$\begin{tikzcd}
(\mathcal{B}_2(\mathbb{D}\times\mathbb{D}), \bullet_\lambda) \arrow[rr, "\mathcal{F}_\lambda"] \arrow[dr, shift right, "\pi"'] & & (L^\infty(\mathcal{H}), \circ) \arrow[dl, "\langle\cdot\rangle_2"] \\
& (\mathcal{B}^2(\mathbb{D}\times\mathbb{D}), *_\lambda) \arrow[ul, hook, shift right, "\iota"'] \arrow[d, shift right, "\delta^*"'] & \\
& (\mathcal{B}(\mathbb{D}), *_\lambda) \arrow[u, shift right, "c"'] &
\end{tikzcd} \tag{4.18}$$

The notions of covariant and contravariant symbols were introduced by Berezin and their importance in various aspects of quantization was shown in [20,21]. The two-contravariant and two-covariant symbols of Schatten class operators and bounded operators were introduced and studied in [22].

In the following proposition we will mention a few properties of the quantum scalar conformal phase space $\mathcal{M}^{++}$ and its C^*-subalgebra $\mathcal{M}^{++}_{\mathrm{pol}}$.

Proposition 5.

(i) *The autorepresentation of* $\mathcal{M}^{++}_{\mathrm{pol}}$ *in* $L^\infty(\mathcal{H})$ *is irreducible and* $\mathcal{P}^{++}_{\mathrm{pol}} \cap \overline{\mathcal{P}^{++}_{\mathrm{pol}}} = \mathbb{C}\mathbb{I}$.
(ii) $\mathcal{M}^{++}_{\mathrm{pol}}$ *is weakly* (*strongly*) *dense in* $L^\infty(\mathcal{H})$.
(iii) $\mathcal{M}^{++}_{\mathrm{pol}}$ *contains the ideal* $L^0(\mathcal{H})$ *of compact operators. Thus any ideal of* $\mathcal{M}^{++}_{\mathrm{pol}}$, *which autorepresentation in* $\mathcal{H}$ *is irreducible*, *also contains* $L^0(\mathcal{H})$.
(iv) $\mathcal{M}^{++}_{\mathrm{pol}}$ *is conformally invariant*, *i.e.* $U_\lambda(g)\mathcal{M}^{++}_{\mathrm{pol}}U_\lambda(g)^\dagger \subset \mathcal{M}^{++}_{\mathrm{pol}}$ *for* $g \in \mathrm{SU}(2,2)$.
(v) $\mathcal{P}^{++}_{\mathrm{pol}} \cap L^0(\mathcal{H}) = \{0\}$.
(vi) $L^0(\mathcal{H}) \subsetneq Comm\mathcal{M}^{++}_{\mathrm{pol}}$, *where* $Comm\mathcal{M}^{++}_{\mathrm{pol}}$ *is commutator ideal of* $\mathcal{M}^{++}_{\mathrm{pol}}$.
(vi) *The statements* (*i*), (*ii*), (*iii*), (*v*), and (*vi*) *are valid also for* $\mathcal{M}^{++}$ *and* $\mathcal{P}^{++}$.

Proof.

(i) Let us denote by P the orthogonal projector defined by decomposition of $\mathcal{H}$ on the Hilbert subspaces irreducible with respect to $\mathcal{M}^{++}_{\mathrm{pol}}$. Let us define $p \in L^2\mathcal{O}(\mathbb{D}, \mathrm{d}\mu_\lambda)$ by

$$p(Z) := \overline{\left\langle Z \,\middle|\, P \,\middle|\, \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \right\rangle}. \tag{4.19}$$

Since

$$a(f)^{\dagger} P = Pa(f)^{\dagger} \tag{4.20}$$

for each $f \in Pol(\bar{\mathbb{D}})$ then from (3.38) and (4.20) one has

$$(I_\lambda \circ P \circ I_\lambda^{-1}) I_\lambda \left(\left| \begin{matrix} j & m \\ j_1 & j_2 \end{matrix} \right\rangle \right) = p I_\lambda \left(\left| \begin{matrix} j & m \\ j_1 & j_2 \end{matrix} \right\rangle \right). \tag{4.21}$$

Let us observe that $I_\lambda \circ P \circ I_\lambda^{-1}$ is the operator of multiplication by the holomorphic function p and $P^2 = P \neq 0$ and $\mathbb{D}$ is connected set. Thus $p \equiv 1$ and from (4.21) we obtain that $P = \mathbb{I}$, what proves irreducibility of $\mathcal{M}_{\mathrm{pol}}^{++}$. If $x \in \mathcal{P}_{\mathrm{pol}}^{++} \cap \overline{\mathcal{P}_{\mathrm{pol}}^{++}}$ then it commutes with any element of $\mathcal{M}_{\mathrm{pol}}^{++}$. So $x \in \mathbb{C}\mathbb{I}$.

(ii) It follows from (i) and from the von Neumann bicommutant theorem.

(iii) Let us take the operator $F \in L^{\infty}(\mathcal{H})$ which has finite number of nonzero matrix elements in the orthonormal basis (3.7). Then its two-covariant symbol is given by

$$\langle F \rangle_2 (Z^{\dagger}, V) = \sum_{\substack{(j, m, j_1, j_2) \in \Phi \\ (j', m', j_1', j_2') \in \Phi}} (\det(E - Z^{\dagger} V)^{\lambda} \Delta_{j_1 j_2}^{jm}(Z^{\dagger})) \times \left\langle \begin{matrix} j & m \\ j_1 & j_2 \end{matrix} \right| F \left| \begin{matrix} j' & m' \\ j_1' & j_2' \end{matrix} \right\rangle \Delta_{j_1' j_2'}^{j'm'}(V), \tag{4.22}$$

where Φ is a finite index set. The operator

$$\sum_{\substack{(j,m,j_1,j_2) \in \Phi \\ (j',m',j_1',j_2') \in \Phi}} \Delta_{j_1 j_2}^{jm}(\mathbb{A}^{\dagger}) \left\langle \begin{matrix} j & m \\ j_1 & j_2 \end{matrix} \right| F \left| \begin{matrix} j' & m' \\ j_1' & j_2' \end{matrix} \right\rangle \Delta_{j_1' j_2'}^{j'm'}(\mathbb{A}) \tag{4.23}$$

belongs to $\mathcal{M}_{\mathrm{pol}}^{++}$ and has the same two-covariant symbol as operator F. Thus we gather that finite rank operator F is equal to (4.23) what implies that $F \in \mathcal{M}_{\mathrm{pol}}^{++}$. So from the fact that $L^0(\mathcal{H}) \cap \mathcal{M}_{\mathrm{pol}}^{++} \neq \{0\}$ and Theorem 2.4.9 in [23] we see that $L^0(\mathcal{H}) \subset \mathcal{M}_{\mathrm{pol}}^{++}$.

(iv) Since $\mathcal{M}_{\mathrm{pol}}^{++}$ is generated by $\mathcal{P}_{\mathrm{pol}}^{++}$, the statement follows from Proposition 3.

(v) Let $f \in C(\bar{\mathbb{D}})$ and $(\mathcal{F}_\lambda \circ \iota \circ c)(f)$ belongs to $L^0(\mathcal{H})$ and $\mathcal{P}_{\mathrm{pol}}^{++}$ then its spectrum is discrete and equal to $f(\bar{\mathbb{D}})$ at the same time, which leads to a contradiction.

(vi) From (iii) one has that $|\phi\rangle\langle\psi| \in \mathcal{M}_{\mathrm{pol}}^{++}$ for $\phi, \psi \in \mathcal{H}$. Additionally one has

$$|\phi\rangle\langle\psi| = (|\phi\rangle\langle v|)(|v\rangle\langle\psi|), \tag{4.24}$$

$$|\phi\rangle\langle v| = [|\phi\rangle\langle\eta|, |\eta\rangle\langle v|] \tag{4.25}$$

if $v, \eta \in \mathcal{H}$ satisfy $\langle v|v\rangle = \langle\eta|\eta\rangle = 1$ and $\langle v|\psi\rangle = 0$. Hence $L^0(\mathcal{H}) \subset \mathrm{Comm}\, \mathcal{M}_{\mathrm{pol}}^{++}$.

In order to show that $L^0(\mathcal{H}) \subsetneq \mathrm{Comm}\,\mathcal{M}^{++}_{\mathrm{pol}}$ we observe that the operator $[a^\dagger_{11}, a_{11}] \in \mathrm{Comm}\,\mathcal{M}^{++}_{\mathrm{pol}}$ in the basis (3.7) assumes the form

$$[a^\dagger_{11}, a_{11}]\left|\begin{matrix} j & m \\ j_1 & j_2 \end{matrix}\right\rangle = \frac{\begin{matrix}(\lambda-2)((j_1+j_2)(m+2j+\lambda)-(m+2j+\lambda)(m+\lambda-2)\\ -(j+j_1+1)(j+j_2+1))\end{matrix}}{(m+2j+\lambda-1)(m+2j+\lambda)(m+\lambda-2)(m+\lambda-1)}\left|\begin{matrix} j & m \\ j_1 & j_2 \end{matrix}\right\rangle. \tag{4.26}$$

Thus it is diagonal and $\frac{1}{4}\frac{2-\lambda}{(m+\lambda-2)(m+\lambda-1)}$ is the concentration point of its spectrum. So, it is not compact operator.

(vii) It follows from the fact that $\mathcal{P}^{++}_{\mathrm{pol}} \subset \mathcal{P}^{++}$. □

Now let us make few remarks about the Toeplitz (holomorphic) representation of $\mathcal{M}^{++}$, i.e. the representation in the Hilbert space $L^2\mathcal{O}(\mathbb{D}, \mathrm{d}\mu_\lambda)$. One obtains it using the anti-linear monomorphism $I_\lambda : \mathcal{H} \to L^2\mathcal{O}(\mathbb{D}, \mathrm{d}\mu_\lambda)$ given by (3.16):

$$\mathcal{T}_\lambda(X) := I_\lambda \circ X \circ I_\lambda^{-1} : L^2\mathcal{O}(\mathbb{D}, \mathrm{d}\mu_\lambda) \to L^2\mathcal{O}(\mathbb{D}, \mathrm{d}\mu_\lambda), \tag{4.27}$$

where $X \in \mathcal{M}^{++}$. In the particular case when $X \in \overline{\mathcal{P}^{++}}$ one has

$$\mathcal{T}_\lambda(X)\psi(Z) = \langle X\rangle(Z)\psi(Z). \tag{4.28}$$

So, $\mathcal{T}(\overline{\mathcal{P}^{++}_{\mathrm{pol}}})$ is realized by multiplication operators M_f, $f \in H^\infty(\mathbb{D})$, having a continuous prolongation to $\overline{\mathbb{D}}$. Thus, the Toeplitz algebra $\mathcal{T}_\lambda(\mathcal{M}^{++}_{\mathrm{pol}})$ is generated by the operators

$$\mathcal{T}_\lambda(f) = \Pi_\lambda \circ M_f \circ \Pi_\lambda, \tag{4.29}$$

where f is a real analytic polynomial. The operator $M_f : L^2\mathcal{O}(\mathbb{D}, \mathrm{d}\mu_\lambda) \to L^2\mathcal{O}(\mathbb{D}, \mathrm{d}\mu_\lambda)$ is the operator of multiplication by $f \in C(\overline{\mathbb{D}})$ and

$$(\Pi_\lambda\psi)(Z) = \int_{\mathbb{D}} \overline{\langle Z|V\rangle}\psi(V^\dagger, V)\frac{1}{\langle V|V\rangle}c_\lambda \mathrm{d}\mu(V^\dagger, V) \tag{4.30}$$

is the orthogonal projector Π_λ of the Hilbert space $L^2(\mathbb{D}, \mathrm{d}\mu_\lambda)$ on its Hilbert subspace $L^2\mathcal{O}(\mathbb{D}, \mathrm{d}\mu_\lambda)$.

Using the representation (4.27) one can investigate $\mathcal{M}^{++}_{\mathrm{pol}}$ in the framework of theory of Toeplitz algebras related to bounded symmetric domains, which were intensively investigated in series of works [24–26]. scalar phase space $\mathcal{M}^{++}$.

The following basic statement can be viewed as a variant of the Coburn Theorem (see [27]).

Theorem 6. *One has the exact sequence*

$$0 \longrightarrow \mathrm{Comm}\,\mathcal{M}^{++}_{\mathrm{pol}} \overset{\iota}{\to} \mathcal{M}^{++}_{\mathrm{pol}} \overset{\pi_\lambda}{\to} C(\mathbb{M}^{00}) \longrightarrow 0 \tag{4.31}$$

of C^*-algebra homomorphisms, where $C(\mathbb{M}^{00})$ is the C^*-algebra of continuous functions on the conformally compactified Minkowski space $\mathbb{M}^{00}$.

Proof. We begin observing that for $f \in C(\bar{\mathbb{D}})$ one has inequalities

$$\|\mathcal{T}_\lambda(f)\|_\infty \le \|f\|_{\sup} \le \|\mathcal{Q}_\lambda(f)\|_\infty, \tag{4.32}$$

which follow from (4.29) and from (5.5) respectively. From the first inequality in (4.32) it follows that the map

$$C(\bar{\mathbb{D}}) \ni f \longrightarrow T_\lambda(f) := [\mathcal{T}_\lambda(f)] \in \mathcal{M}^{++}_{\mathrm{pol}}/\mathrm{Comm}\,\mathcal{M}^{++}_{\mathrm{pol}} \tag{4.33}$$

is a continuous epimorphism of the C^*-algebra $C(\bar{\mathbb{D}})$ on the commutative quotient C^*-algebra $\mathcal{M}^{++}_{\mathrm{pol}}/\mathrm{Comm}\,\mathcal{M}^{++}_{\mathrm{pol}}$. Let us recall that the norm of $[x] \in \mathcal{M}^{++}_{\mathrm{pol}}/\mathrm{Comm}\,\mathcal{M}^{++}_{\mathrm{pol}}$ is defined by

$$\|[x]\|_{inf} = \inf_{\xi \in \mathrm{Comm}\,\mathcal{M}^{++}_{\mathrm{pol}}} \|x + \xi\|. \tag{4.34}$$

Now let us consider the ideal $\ker T_\lambda \subset C(\bar{\mathbb{D}})$. It follows from (iv) of Proposition 5 that $U_\lambda(g)(\mathrm{Comm}\,\mathcal{M}^{++}_{\mathrm{pol}})U_\lambda(g)^\dagger \subset \mathrm{Comm}\,\mathcal{M}^{++}_{\mathrm{pol}}$, so the conformal group $\mathrm{SU}(2,2)/\mathbb{Z}_4$ acts on the quotient C^*-algebra $\mathcal{M}^{++}_{\mathrm{pol}}/\mathrm{Comm}\,\mathcal{M}^{++}_{\mathrm{pol}}$ and the C^*-algebra epimorphism defined by (4.33) is a conformally equivariant map, i.e.

$$\begin{tikzcd} C(\bar{\mathbb{D}}) \arrow[r, "T_\lambda"] \arrow[d, "\Sigma_g"'] & \mathcal{M}^{++}_{pol}/Comm\mathcal{M}^{++}_{pol} \arrow[d, "{[U_\lambda(g)]}"] \\ C(\bar{\mathbb{D}}) \arrow[r, "T_\lambda"'] & \mathcal{M}^{++}_{pol}/Comm\mathcal{M}^{++}_{pol} \end{tikzcd} \tag{4.35}$$

for any $g \in \mathrm{SU}(2,2)/\mathbb{Z}_4$, where

$$(\Sigma_g f)(Z^\dagger, Z) := f((\sigma_g(Z))^\dagger, \sigma_g(Z)) \tag{4.36}$$

$$[U_\lambda(g)]([x]) := [U_\lambda(g) x U_\lambda(g)^\dagger]. \tag{4.37}$$

We conclude from the above that $\ker T_\lambda$ is an ideal in $C(\bar{\mathbb{D}})$ conformally invariant with respect to the action (4.36). Since any ideal in $C(\bar{\mathbb{D}})$ consists of functions vanishing on some compact subset $K \subset \bar{\mathbb{D}}$ the conformally invariant ideals correspond to the conformally invariant compact subsets: $\bar{\mathbb{D}}$, $\partial\mathbb{D} = \{Z \in \mathrm{Mat}_{2\times 2}(\mathbb{C}) : \det(E - Z^\dagger Z) = 0 \text{ and } \mathrm{Tr}(E - Z^\dagger Z) \ge 0\}$ and $U(2) = \{Z \in \mathrm{Mat}_{2\times 2}(\mathbb{C}) : Z^\dagger Z = E\}$, where the last one is the Šilov boundary of $\mathbb{D}$. In this way we show that $\ker T_\lambda$ is equal to one of the following three ideals

$$\mathcal{I}_{\bar{\mathbb{D}}} = \{0\} \subset \mathcal{I}_{\partial\bar{\mathbb{D}}} \subset \mathcal{I}_{U(2)}, \tag{4.38}$$

where by $\mathcal{I}_K$ we denote the ideal of functions equal to zero on K. The polynomial

$$\phi(Z^\dagger, Z) := \mathrm{Tr}(E - Z^\dagger Z) \tag{4.39}$$

generates the ideal $\mathcal{I}_{U(2)}$ and maps $\bar{\mathbb{D}}$ on the interval $[0, 2]$. Let us consider the positive operator

$$: \mathrm{Tr}(E - \mathbb{A}^\dagger \mathbb{A}) := 2 - a^\dagger_{11}a_{11} - a^\dagger_{12}a_{12} - a^\dagger_{21}a_{21} - a^\dagger_{22}a_{22}, \tag{4.40}$$

which is diagonal, with

$$: \mathrm{Tr}(E - \mathbb{A}^\dagger \mathbb{A}) : \left| \begin{matrix} j & m \\ j_1 & j_2 \end{matrix} \right\rangle = \frac{2(\lambda - 2)(m + j + \lambda - 1)}{(m + \lambda - 1)(m + 2j + \lambda)} \left| \begin{matrix} j & m \\ j_1 & j_2 \end{matrix} \right\rangle, \tag{4.41}$$

in the basis (3.7). We see from (4.41) that the spectrum σ of $: \mathrm{Tr}(E - \mathbb{A}^\dagger\mathbb{A}) :$ is contained in the interval $[0, 2]$ and the set

$$\sigma_a := \left\{ \frac{\lambda - 2}{m + \lambda - 1} : m \in \mathbb{N} \cup \{0\} \cup \{\infty\} \right\} \tag{4.42}$$

is its approximative spectrum. The continuous function $F : [0, 2] \to \mathbb{R}$ defined by

$$F(x) := x \sin \frac{(\lambda - 2)\pi}{x}, \tag{4.43}$$

vanishes on σ_a and $F \circ \phi \in \mathcal{I}_{U(2)}$. Since $F_{|\sigma_a} \equiv 0$ and F assumes the same value at most on a finite subset of $\sigma \setminus \sigma_a$, we conclude that $F(: \mathrm{Tr}(E - \mathbb{A}^\dagger\mathbb{A}) :)$ is a compact operator. Thus, by (iii) of Proposition 5 $F(: \mathrm{Tr}(E - \mathbb{A}^\dagger\mathbb{A}) :)$ belongs to $\mathrm{Comm}\,\mathcal{M}^{++}_{\mathrm{pol}}$. Let us take the sequence $\{P_n(x)\}_{n\in\mathbb{N}}$ of polynomials which uniformly approximate $P_n \to F$ the function $F \in C([0, 2])$. Thus one has

$$\|P_n \circ \phi - F \circ \phi\|_{\sup} \underset{n\to\infty}{\to} 0. \tag{4.44}$$

From (4.44) and the first inequality of (4.32) we obtain

$$\|\mathcal{T}_\lambda(P_n \circ \phi) - \mathcal{T}_\lambda(F \circ \phi)\|_\infty \underset{n\to\infty}{\to} 0. \tag{4.45}$$

On the other hand, from the Gelfand–Naimark theorem and (4.44) we have

$$\|P_n(: \mathrm{Tr}(E - \mathbb{A}^\dagger\mathbb{A}) :) - F(: \mathrm{Tr}(E - \mathbb{A}^\dagger\mathbb{A}) :)\|_\infty \underset{n\to\infty}{\to} 0. \tag{4.46}$$

The operators $\mathcal{T}_\lambda(P_n \circ \phi)$ are polynomials of the creation and annihilation operators taken in the anti-normal ordering, so they differ from the polynomials $P_n(: \mathrm{Tr}(E - \mathbb{A}^\dagger\mathbb{A}) :)$ modulo elements of $\mathrm{Comm}\,\mathcal{M}^{++}_{\mathrm{pol}}$. Thus, using also (4.45) and (4.46), we obtain that

$$0 = \|[\mathcal{T}_\lambda(F \circ \phi)] - [F(: \mathrm{Tr}(E - \mathbb{A}^\dagger\mathbb{A}) :)]\|_{\inf} = \|[\mathcal{T}_\lambda(F \circ \phi)]\|_{\inf} = \|T_\lambda(F \circ \phi)\|_{\inf}. \tag{4.47}$$

Summing up we conclude that $F \circ \phi \in \ker T_\lambda \cap \mathcal{I}_{U(2)}$. Since it is easy to check that $F \circ \phi \notin \mathcal{I}_{\partial\mathbb{D}}$ and that $\ker T_\lambda$, is conformally invariant it follows that $\ker T_\lambda = \mathcal{I}_{U(2)} = \mathcal{I}_{\mathbb{M}^{00}}$.

Taking into account that (4.33) is an epimorphism of C^*-algebras, we state the following isomorphisms $\mathcal{M}^{++}_{\mathrm{pol}}/\mathrm{Comm}\,\mathcal{M}^{++}_{\mathrm{pol}} \cong C(\bar{\mathbb{D}})/\mathcal{I}_{\mathbb{M}^{00}} \cong C(\mathbb{M}^{00})$. These isomorphisms give the epimorphism $\pi_\lambda : \mathcal{M}^{++}_{\mathrm{pol}} \to C(\mathbb{M}^{00})$. □

Ending this section, let us remark that "neglecting" the non-commutativity of quantum complex Minkowski space $\mathcal{M}^{++}_{\mathrm{pol}}$ we come back to the commutative C^*-algebra $C(\mathbb{M}^{00})$ whose spectrum is given by the conformally compactified Minkowski space $\mathbb{M}^{00}$.

5. Quantization and physical interpretation

Analogously to the classical coordinate observables $(Z, Z^\dagger)$ on $\mathbb{M}^{++}$ we shall use quantum coordinate observables $(\mathbb{A}, \mathbb{A}^\dagger)$ for the quantum phase space $\mathcal{M}^{++}$. Superposing morphisms from diagram (4.18) we obtain the extension of this correspondence. In such a way we get the isomorphism

$$Q_\lambda := \mathcal{F}_\lambda \circ \iota \circ c : \mathcal{B}(\mathbb{D}) \longrightarrow L^\infty(\mathcal{H}), \tag{5.1}$$

which extends the quantization map,

$$a : H^\infty(\mathbb{D}) \ni f \longrightarrow a(f) \in L^\infty(\mathcal{H}), \tag{5.2}$$

discussed in the previous section. Taking into account the properties

$$Q_\lambda(f *_\lambda g) = Q_\lambda(f)Q_\lambda(g), \tag{5.3}$$

$$Q_\lambda(\bar{f}) = Q_\lambda(f)^*, \tag{5.4}$$

$$\langle Q_\lambda(f)\rangle_\lambda = f, \tag{5.5}$$

for $f, g \in \mathcal{B}(\mathbb{D})$, we see that the isomorphism Q_λ gives a *quantization procedure* inverse to the mean value map.

According to relation (5.5), Berezin covariant symbols are the classical observables corresponding to the quantum observables realized by the bounded operators. As a particular case the quantum phase space $\mathcal{M}^{++} \subset L^\infty(\mathcal{H})$ is obtained from $\langle\mathcal{M}^{++}\rangle \subset \mathcal{B}(\mathbb{D})$ by the quantization (5.1). However for physical reasons we are interested in the extension of $Q_\lambda : \mathcal{B}(\mathbb{D}) \to L^\infty(\mathcal{H})$ to a larger algebra of observables. For example it is reasonable to include in this scheme the elements of the enveloping algebra of the conformal Lie algebra su(2, 2). The latter ones are represented by unbounded operators in $\mathcal{H}$ which, according to the equivariance property (4.35), possess the common domain given by the linear span $\mathcal{L}(\mathcal{K}_\lambda(\mathbb{M}^{++}))$ of the set $\mathcal{K}_\lambda(\mathbb{M}^{++})$ of the coherent states. Let us then define the vector space $\mathcal{A}^{++}$ of operators in $\mathcal{H}$ closed with respect to the operation of conjugation and all elements of which possess $\mathcal{L}(\mathcal{K}_\lambda(\mathbb{M}^{++}))$ as a common domain. Therefore for any operator $F \in \mathcal{A}^{++}$ the two-covariant and Berezin covariant symbols have sense.

In the following we will use the coherent state weak topology, i.e. $A_n \overset{\text{coh}}{\to} A$ if $\langle Z|A_n|V\rangle \to \langle Z|A|V\rangle$ for all $Z, V \in \mathbb{D}$. It is a weaker topology than the weak one, as can be seen from the following example. Let $\mathbb{D} \ni Z_n = (1 - \frac{1}{n})E$, $n \in \mathbb{N}$. We define the sequence of operators

$$A_n := n\frac{|Z_n\rangle\langle Z_n|}{\langle Z_n|Z_n\rangle}. \tag{5.6}$$

It is easily observed that

$$\forall\, Z, V \in \mathbb{D} \qquad \lim_{n\to\infty}\langle Z|A_n|V\rangle = 0, \tag{5.7}$$

thus $A_n \overset{\text{coh}}{\to} 0$. On the other hand $\sup_{n\in\mathbb{N}}\|A_n\| = \infty$, thus A_n is not weakly convergent.

The space $\mathcal{A}^{++}$ is closed with respect to coherent state weak topology. The quantum phase space $\mathcal{M}^{++}$ is contained in $\mathcal{A}^{++}$ as a dense subset with respect to the coherent state weak topology. For any $F \in \mathcal{A}^{++}$ its Berezin symbol $f = \langle F\rangle \in \mathcal{RO}^{++}(\mathbb{D})$ is the real analytic function

$$f(Z^\dagger, Z) = \sum f_{i_{11},i_{12},i_{21},i_{22},j_{11},j_{12},j_{21},j_{22}}\bar{Z}_{11}^{i_{11}}\bar{Z}_{12}^{i_{12}}\bar{Z}_{21}^{i_{21}}\bar{Z}_{22}^{i_{22}}Z_{11}^{j_{11}}Z_{12}^{j_{12}}Z_{21}^{j_{21}}Z_{22}^{j_{22}} \tag{5.8}$$

of the variables $(Z^\dagger, Z)$. One extends the quantization (5.1) naturally to the space $\mathcal{RO}^{++}(\mathbb{D})$ of real analytic functions on $\mathbb{D}$ by setting

$$\begin{aligned} Q_\lambda(f) &= \sum f_{i_{11},i_{12},i_{21},i_{22},j_{11},j_{12},j_{21},j_{22}}a_{11}^{\dagger}{}^{i_{11}}a_{12}^{\dagger}i_{12}a_{21}^{\dagger}i_{21}a_{22}^{\dagger}i_{22}a_{11}i_{11}a_{12}^{i_{12}}a_{21}^{i_{21}}a_{22}^{i_{22}} \\ &=: f(\mathbb{A}^\dagger, \mathbb{A}) :, \end{aligned} \tag{5.9}$$

where as usual, the colons $: \cdot :$ denote normal ordering. The infinite sum in (5.9) is taken in the sense of coherent state weak topology. The extension of the product $*_\lambda$, see (4.16), to the real analytic Berezin symbols $f, g \in \mathcal{RO}^{++}(\mathbb{D})$ is defined by

$$(f *_\lambda g)(Z^\dagger, Z) := \frac{\langle Z^\dagger| : f(\mathbb{A}^\dagger, \mathbb{A}) :: g(\mathbb{A}^\dagger, \mathbb{A}) : |Z\rangle}{\langle Z^\dagger|Z\rangle}. \tag{5.10}$$

As an illustration let us consider the Berezin symbols

$$\langle U_\lambda(g)\rangle(Z^\dagger, Z) = (\det(CZ+D))^{-\lambda}\left(\frac{\det(E - Z^\dagger\sigma_g(Z))}{\det(E - Z^\dagger Z)}\right)^{-\lambda} \tag{5.11}$$

and their quantum $(\mathbb{A}^\dagger, \mathbb{A})$-coordinate representation

$$U_\lambda(g) = Q_\lambda(\langle U_\lambda(g)\rangle) =: \left(\frac{\det(E - \mathbb{A}^\dagger\sigma_g(\mathbb{A}))}{\det(E - \mathbb{A}^\dagger\mathbb{A})}\right)^{-\lambda} : (\det(C\mathbb{A}+D))^{-\lambda} \tag{5.12}$$

for the conformal group elements $g \in \mathrm{SU}(2, 2)$. In order to express the quantum four-momentum, relativistic angular momentum, dilation and four-acceleration in terms of quantum coordinates $(\mathbb{A}^\dagger, \mathbb{A})$ we differentiate $U_\lambda(g(t))$ given by (5.12) with respect to the parameter $t \in \mathbb{R}$ for an appropriate choice of one-parameter subgroup $\mathbb{R} \ni t \to g(t) \in \mathrm{SU}(2, 2)$. As a result one obtains

$$Q_\lambda(p_\mu) = i\lambda : (\det(\mathbb{W} - \mathbb{W}^\dagger))^{-1}\mathrm{Tr}(\sigma_\mu(\mathbb{W} - \mathbb{W}^\dagger)) :, \tag{5.13}$$

$$\begin{aligned} Q_\lambda(m_{\mu\nu}) = i\lambda \Bigg(&\frac{1}{2}\mathrm{Tr}(\sigma_\mu\mathbb{W}^\dagger) : (\det(\mathbb{W} - \mathbb{W}^\dagger))^{-1}\mathrm{Tr}(\sigma_\nu(\mathbb{W} - \mathbb{W}^\dagger)) : \\ &- \frac{1}{2}\mathrm{Tr}(\sigma_\nu\mathbb{W}^\dagger) : (\det(\mathbb{W} - \mathbb{W}^\dagger))^{-1}\mathrm{Tr}(\sigma_\mu(\mathbb{W} - \mathbb{W}^\dagger)) :\Bigg), \end{aligned} \tag{5.14}$$

$$Q_\lambda(d) = i\lambda\mathrm{Tr}(\sigma_\mu\mathbb{W}^\dagger) : (\det(\mathbb{W} - \mathbb{W}^\dagger))^{-1}\mathrm{Tr}(\sigma^\mu(\mathbb{W} - \mathbb{W}^\dagger)) : -2i\lambda\mathbb{I}, \tag{5.15}$$

$$\begin{aligned} Q_\lambda(a_\nu) = i\lambda \det(\mathbb{W}^\dagger) &: (\det(\mathbb{W} - \mathbb{W}^\dagger))^{-1}\mathrm{Tr}(\sigma_\nu(\mathbb{W} - \mathbb{W}^\dagger)) : \\ &- i\lambda\frac{1}{2}\mathrm{Tr}(\sigma_\nu\mathbb{W}^\dagger)\mathrm{Tr}(\sigma^\beta\mathbb{W}^\dagger) : (\det(\mathbb{W} - \mathbb{W}^\dagger))^{-1}\mathrm{Tr}(\sigma_\beta(\mathbb{W} - \mathbb{W}^\dagger)) : \\ &+ + i\lambda\mathrm{Tr}(\sigma_\nu\mathbb{W}^\dagger), \end{aligned} \tag{5.16}$$

where $(\mathbb{W}^\dagger, \mathbb{W})$ are matrix operator coordinates in $\mathcal{A}^{++}$ obtained from $(\mathbb{A}^\dagger, \mathbb{A})$ by the Caley transform

$$\mathbb{W} = i(\mathbb{A} + E)(\mathbb{A} - E)^{-1}, \tag{5.17}$$

which has sense in the coherent state weak topology. After passing to the representation in the Hilbert space $L^2\mathcal{O}(\mathbb{T}, \mathrm{d}\mu_\mu)$ of holomorphic functions on the future tube $\mathbb{T}$, square integrable with respect to the measure (3.24), we rediscover from (5.13)–(5.16) the operators (3.20)–(3.23) obtained by the Kostant–Souriau geometric quantization.

It follows from (3.20) that

$$[Q_\lambda(p_\mu), Q_\lambda(p_\nu)] = 0. \tag{5.18}$$

Using (2.21), we see from (5.18) that

$$[Q_\lambda(y_\mu), Q_\lambda(y_\nu)] = 0. \tag{5.19}$$

The creation operators

$$Q_\lambda(\bar{w}^\mu) = \tfrac{1}{2}\mathrm{Tr}(\sigma_\mu\mathbb{W}^\dagger) \tag{5.20}$$

in $L^2\mathcal{O}(\mathbb{T}, d\mu_\lambda)$ are given as multiplication by the complex coordinate functions w^μ, so they commute. Thus, in addition to (5.18), we have

$$[Q_\lambda(x^\mu), Q_\lambda(x^\nu)] = 0, \tag{5.21}$$

$$[Q_\lambda(x^\mu), Q_\lambda(p_\nu)] = -i\delta^\mu_\nu 1 \tag{5.22}$$

for the quantum canonical coordinates $(Q_\lambda(x^\mu), Q_\lambda(p_\nu))$.

Therefore we see that Heisenberg algebra generated by unbounded operators of four-momenta $Q_\lambda(p_\nu)$ and four-positions $Q_\lambda(x^\mu) = \frac{1}{2}\mathrm{Tr}(\sigma_\mu(\mathbb{W} + \mathbb{W}^\dagger))$ is included in $\mathcal{A}^{++}$. The creation operators (5.20) and the annihilation ones

$$Q_\lambda(w_\nu) = \tfrac{1}{2}\mathrm{Tr}(\sigma_\nu \mathbb{W}), \tag{5.23}$$

generate the Caley transforms of quantum polarizations $\overline{\mathcal{P}^{++}_{\mathrm{pol}}}$ and $\mathcal{P}^{++}_{\mathrm{pol}}$ respectively. However their commutators $[Q_\lambda(\bar{w}^\mu), Q_\lambda(w_\nu)] \neq 0$ do not have so simple form as it has place in the case of quantum real polarization given by the canonical commutation relation (5.22).

Let us now discuss the physical sense of the parameter $\lambda \in \mathbb{R}$. So far, for technical reasons, we assumed that it was dimensionless. However, as one sees from (2.21), λ has dimensions of action. We therefore assume the Planck constant h as the natural unit for λ. After this we obtain

$$w^\mu = x^\mu + i\lambda \frac{h}{mc}\frac{p^\mu}{mc}, \tag{5.24}$$

where $mc = \sqrt{p_0^2 - \vec{p}^2}$. The quantity $\frac{h}{mc}$ is the Compton wavelength of the conformal particle. For example for the proton $\frac{h}{mc} \cong 10^{-13}$ cm.

The quantities $\frac{p^\mu}{mc}$ denote the components of relativistic four-velocity measured with the speed of light as the unit. Dimensional analysis shows that in the limit $\lambda \to \infty$ the theory describes physical phenomena characterized by a space-time scale much bigger than the Compton scale characteristic for the quantum phenomena. This physical argument is consistent with the following asymptotic behavior of $*_\lambda$-product

$$f *_\lambda g \sim fg, \tag{5.25}$$

$$f *_\lambda g - g *_\lambda f \sim i\lambda\{f, g\} \tag{5.26}$$

for $\lambda \to \infty$, where the right hand side of (5.25) is usual multiplication of functions and the right side of (5.26) is the Poisson bracket (3.19). In order to show these asymptotic formulae we apply the method used for the case of a general symmetric domain in [20]. The expressions (5.25), (5.26) show the correspondences of the quantum description of the massive scalar conformal particle to its classical mechanical description in the large space-time scale limit.

The quantum effects are described by the transition amplitude (4.17), which in the coordinates $(\bar{w}^\mu, w^\nu)$ is given by

$$a_\lambda(v^\dagger, w) = \left(\frac{((w-\bar{w})^2(v-\bar{v})^2)^{1/2}}{(w-\bar{v})^2}\right)^\lambda, \tag{5.27}$$

where $(w-\bar{v})^2 = \eta_{\mu\nu}(w^\mu - \bar{v}^\mu)(w^\nu - \bar{v}^\nu)$ and $\lambda > 3$. One sees from (5.27) that the transition probability $\left|a_\lambda(v^\dagger, w)\right|^2$ from w to v as a function of v forms a narrow peak around the coherent state $w \in \mathbb{T}$ if $\lambda\frac{h}{mc} \approx 0$. A more detailed physical discussion can be found in [6].

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